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HOLOMORPHIC ANOMALY EQUATION FOR THE HODGE-GROMOV-WITTEN INVARIANTS OF ELLIPTIC CURVES

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Abstract. We study the modularity and holomorphic anomaly equation for Hodge-Gromov-Witten invariants of elliptic curves.

Contents

0. Introduction

0.1. Overview

Let E be an elliptic curve. Denote by

$$\overline{M}_{g,n}(E,d)$$

the moduli space of degree d stable maps of genus g to E with n markings. Let

$$\pi : \overline{M}_{g,n}(E,d) \longrightarrow \overline{M}_{g,n}\,,\ \mathrm{ev}_i : \overline{M}_{g,n}(E,d) \longrightarrow E\,,$$

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be the universal structures. For cohomological classes $\gamma_1, \ldots, \gamma_n \in H^*(E)$, we define Gromov-Witten classes of E by

$$\mathsf{GW}_{g,d}(\gamma_1, \ldots, \gamma_n) := \pi_*\Big([\overline{M}_{g,n}(E,d)]^{\mathrm{vir}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\Big) \in H^*(\overline{M}_{g,n})\,.$$

While the torus localization method introduced in [8] give satisfactory answers to the Gromov-Witten theories of varieties with appropriate torus actions, this is not the case for the Gromov-Witten theories of varieties like E which do not have such torus actions. There have been many other methods to study the Gromov-Witten theory of E. See [6, 18, 19].

Recently there appeared several new techniques which make it possible to study Gromov-Witten theory of E via the torus localization method ([4, 10, 11]). We apply the technique in [10] to study the Gromov-Witten invariants of E via the torus localization method.

0.2. Hodge-Gormov-Witten invariants

Let E be an elliptic curve. Denote by

$$\mathbb{E} \longrightarrow \overline{M}_{g,n}(E,d)$$

the Hodge bundle. For cohomological classes $\gamma_1, \ldots, \gamma_n \in H^*(E)$, we define Hodge-Gromov-Witten classes of E by

$$\mathsf{HGW}_{g,d}(\gamma_1, \ldots, \gamma_n) := \pi_*\Big(e(\mathbb{E}) \cdot [\overline{M}_{g,n}(E,d)]^{\mathrm{vir}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\Big) \in H^*(\overline{M}_{g,n})\,.$$

Define the generating series

$$\mathsf{HGW}_{g}(\gamma_1, \ldots, \gamma_n) := \sum_{d=0}^{\infty} \mathsf{H}_{g,d}(\gamma_1, \ldots, \gamma_n)\, Q^d \in H^*(\overline{M}_{g,n}) \otimes \mathbb{Q}[[Q]]\,.$$

In order to state the holomorphic anomaly equation for the Hodge-Gromov-Witten classes of E, we define the following series in q

$$\begin{aligned} L(q) &= (1-27q)^{-\frac{1}{3}} = 1 + 9q + 162q^2 + \ldots\,, \\ C_0(q) &= q\frac{d}{dq}\Big(\log(q) + 3\sum_{d=1}^{\infty} q^d \frac{(3d-1)!}{(d!)^3}\Big)\,, \qquad (1) \\ X(q) &= \frac{q\frac{d}{dq}C_0}{C_0}\,. \end{aligned}$$

We consider the series (1) as series in Q, via the change of variable

$$Q := q \cdot \mathrm{Exp}\Big(3 \sum_{d=1}^{\infty} q^d \frac{(3d-1)!}{(d!)^3}\Big) .$$

THEOREM 1. *For $\gamma_1, \dots, \gamma_n \in H^*(E)$ we have*

$$\mathsf{HGW}_g(\gamma_1, \dots, \gamma_n) \in H^*(\overline{M}_{g,n}) \otimes \mathbb{Q}[L^{\pm 1}, X] .$$

We consider the natural maps

$$\iota : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n} ,$$

which glue the last two marked points of a single $(n+2)$-pointed curve of genus $g-1$ and

$$j : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n} ,$$

which glues the last markings of separate pointed curves for $n = n_1 + n_2$ and $g = g_1 + g_2$.

THEOREM 2. *For the Hodge-Gromov-Witten series of elliptic curve we have*

$$\begin{aligned} \frac{d}{dX}\mathsf{H}_g(\gamma_1, \dots, \gamma_n) =& \iota_* \mathsf{H}_{g-1}(\gamma_1, \dots, \gamma_n, 1, 1) \\ &+ \sum_{\substack{g = g_1 + g_2 \\ \{1,\dots,n\} = S_1 \sqcup S_2}} j_*\Big(\mathsf{H}_{g_1}(\gamma_{S_1}, q) \boxtimes \mathsf{H}_{g_2}(\gamma_{S_2}, 1)\Big) \\ &- 2 \sum_{i=1}^{n} \Big(\int_E \gamma_i\Big) \psi_i \cdot \mathsf{H}_g(\gamma_1, \dots, \gamma_{i-1}, 1, \gamma_{i+1}, \dots, \gamma_n) , \end{aligned}$$

where $\gamma_{S_i} = (\gamma_k)_{k \in S_i}$ and $1 \in H^*(E)$ is the unit.

The derivation of $\mathsf{H}_g(\gamma_1, \dots, \gamma_n)$ with respect to X in the holomorphic anomaly equation of Theorem 2 is well-defined since

$$\mathsf{H}_g(\gamma_1, \dots, \gamma_n) \in H^*(\overline{M}_{g,n}) \otimes \mathbb{Q}[L^{\pm 1}, X]$$

by Theorem 1.

The ring of quasi-modular form is the free polynomial algebra

$$\mathsf{QMod} = \mathbb{Q}[E_2, E_4, E_6] ,$$

where C_k are the weight k Eisenstein series

$$E_k(q) = -\frac{B_k}{k \cdot k!} + \frac{2}{k!} \sum_{n \geq 1} \sum_{d|n} d^{k-1} q^n \,.$$

Here, B_k are the Bernoulli numbers. The series (1) are related to quasi-modular forms as follows.

Lemma 3 ([1]). *We have*

$$E_2 = \frac{C_0^2}{L^3}\,(12X + 4 - 3L^3)\,,$$
$$E_4 = \frac{C_0^4}{L^6}\,(-8L^3 + 9L^6)\,,$$
$$E_6 = \frac{C_0^6}{L^9}\,(-8L^3 + 36L^6 - 27L^9)\,.$$

Applying the Hodge integral formula in [7] to the result of [18], we obtain the following theorem.

Theorem 4. *For $\gamma_1, \ldots, \gamma_n \in H^*(E)$ we have*

$$\mathsf{HGW}_g(\gamma_1, \ldots, \gamma_n) \in H^*(\overline{M}_{g,n}) \otimes \mathsf{QMod}\,.$$

If we compare Theorem 1 and Theorem 4, Lemma 3 yields some tautological relations on $H^*(\overline{M}_{g,n})$. It is interesting question whether these relations can be obtained from Pixton's relations in [20].

0.3. Acknowledgments

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1. Hodge-Gromov-Witten invariants

We review here Hodge-Gromov-Witten theory for chain polynomials studied in [10].

Let $\mathbb{P}(\underline{w}) = \mathbb{P}(w_1, \ldots, w_N)$ be the weighted projective space with weights $w_1, \ldots, w_N \in \mathbb{N}$. Consider a smooth hypersurface X in $\mathbb{P}(\underline{w})$ of degree m polynomial

$$x_1^{a_1} x_2 + \cdots + x_{N-1}^{a_{N-1}} x_N + x_N^{a_N}\,.$$

Let $T = (\mathbb{C}^*)^N$ act diagonally on the vector space $\mathbb{C}^N$ with weight

$$-t_1, \ldots, -t_N\,.$$

Denote the equivariant virtual fundamental class by

$$[\overline{M}_{g,n}(\mathbb{P}(\underline{w}),d)]^{\mathrm{vir},T} \in H_*^T(\overline{M}_{g,n}(\mathbb{P}(\underline{w}),d))\,.$$

THEOREM 5. [10, Theorem 3.3] *Let* $\gamma_1,\ldots,\gamma_n$ *be ambient cohomology classes on* X, *i.e. pulled back from* $\mathbb{P}(\underline{w})$. *For* g, n, $d \in \mathbb{N}$ *with* $2g-2+n>0$, *we have*

$$\begin{aligned} &e(\mathbb{E}^\vee)\cdot[\overline{M}_{g,N}(X,d)]^{\mathrm{vir}}\cdot\prod_{i=1}^N \mathrm{ev}_i^*(\gamma_i) \\ &= \lim_{t=0}\Big[e_T(\mathbb{E}^\vee)\cdot[\overline{M}_{g,N}(\mathbb{P}(\underline{w}),d)]^{\mathrm{vir},T}\cdot e_T(R\pi_*f^*\mathcal{O}(d))\cdot\prod_{i=1}^N \mathrm{ev}_i^*(\gamma_i)\Big]\end{aligned}$$

In the above equation, the class $e_T(R\pi_*f^*\mathcal{O}(d))$ is defined after localization. We use the specialization

$$t_{j+1} = \prod_{k=1}^{j}(-a_k)t$$

for $0 \le j \le N$ before taking the limit on the right-hand side of the equation.

2. localization

2.1. Overview

We summarize here generating series in q which arise in the genus 0 theory of Gromov-Witten invariants. The series will play an important role in the proof of holomorphic anomaly equation for an elliptic curve.

We fix a torus action $\mathsf{T} = (\mathbb{C}^*)^3$ on $\mathbb{P}^2$ with weights $-\lambda_0,-\lambda_1,-\lambda_2$ on the vector space $\mathbb{C}^3$. The T-weight on the fiber over p_i of the anti-canonical bundle

$$\mathcal{O}_{\mathbb{P}^2}(3)\to\mathbb{P}^2$$

is given by $-3\lambda_i$.

For each T-fixed point $p_i \in \mathbb{P}^2$, define

$$e_i = e(T_{p_i}(\mathbb{P}^2))/(3\lambda_i)\,,$$

where $e(T_{p_i}(\mathbb{P}^2))$ is the equivariant Euler class of the tangent space of $\mathbb{P}^2$ at p_i. Let

$$\phi_i = \frac{\prod_{j\neq i}(H-t_j)}{3t_ie_i}\,,\ \ \phi^i = e_i\phi_i\ \in H_{\mathsf{T}}^*(\mathbb{P}^2) \tag{2}$$

be cycle classes.

Define

$$\mathbb{I}(q,z) := \sum_{d=0}^{\infty} \frac{\prod_{k=1}^{d}(3H+kz)}{\prod_{i=0}^{2}\prod_{k=1}^{d}(H+z-\lambda_i)} \in H^*(\mathbb{P}^2)\otimes\mathbb{Q}[[q]]\,.$$

We define for $i=0,1,2$,

$$\mathbb{S}_i(\gamma) := e_i\langle\langle \frac{\phi_i}{z-\psi}, \gamma\rangle\rangle_{0,2}^{0+}\,.$$

We write

$$\mathbb{S}(\gamma) = \sum_{i=0}^{2}\phi_i\mathbb{S}_i(\gamma)\,.$$

Define series C_0, C_1, C_2 and $N_2(q)$, $N_3(q)$, $N_4(q)$ by the following equations,

$$\begin{aligned}
\mathbb{I} &= C_0 + \mathsf{O}(\frac{1}{z})\,,\\
(H+z\frac{d}{dq})\mathbb{S}(1) &= C_1H + N_2 + \mathsf{O}(\frac{1}{z})\,,\\
(H+z\frac{d}{dq})\mathbb{S}(H) &= C_2H^2 + N_3H + N_4 + \mathsf{O}(\frac{1}{z})\,.
\end{aligned}$$

The following relations were obtained in [21],

$$\begin{aligned}
C_0 &= C_2\,, \\
C_0C_1C_2 &= L^3\,.
\end{aligned} \tag{3}$$

The following equations were proven in [15].

$$\begin{aligned}
&\frac{1}{2}\sum_i t_i(C_1-L^3)+N_2 = 0\,,\\
&\sum_i t_iC_1 + N_2 - \sum_i t_i - N_3 = 0\,,\\
&\Big(\sum_i t_i\Big)^2\frac{(1-C_0^4)L^3}{4C_0^2} - \sum_{i>j}t_it_j(1-C_0)C_0 + N_4 = 0\,.
\end{aligned} \tag{4}$$

2.2. Further calculations

Using Birkhoff factorization, an evaluation of the series $\mathbb{S}(H^j)$ can be obtained from the $\mathbb{I}$-function, see [13]:

$$
\begin{aligned}
\mathbb{S}(1) &= \frac{\mathbb{I}}{C_0}\,, \\
\mathbb{S}(H) &= \frac{\mathsf{M}\,\mathbb{S}(1) - N_2\,\mathbb{S}(1)}{C_1}\,, \\
\mathbb{S}(H^2) &= \frac{\mathsf{M}\,\mathbb{S}(H) - N_3\mathbb{S}(H) - N_4\mathbb{S}(1)}{C_2}\,.
\end{aligned} \tag{5}
$$

Here $\mathsf{M} := H + z\,q\frac{d}{dq}$.

The function $\mathbb{I}$ satisfies following Picard-Fuchs equation

$$
\Big(\prod_{i=0}^{2}(\mathsf{M} - \lambda_i) - q(3\mathsf{M})(3\mathsf{M}+z)(3\mathsf{M}+2z)\Big)\mathbb{I} = 0\,. \tag{6}
$$

The restriction $\mathbb{I}|_{H=\lambda_i}$ admits following asymptotic expansion

$$
\mathbb{I}|_{H=\lambda_i} = e^{\mu_i/z}\Big(R_{i,0} + R_{i,1}z + R_{i,2}z^2 + \dots\Big)\,. \tag{7}
$$

The series μ_i and $R_{i,k}$ can be explicitly calculated by solving differential equations obtained from the coefficient of z^k in the Picard-Fuchs equation (6),

$$
\begin{aligned}
\mu_i(q) =& \int_0^q \frac{L_i(x) - t_i}{x} dx\,, \\
R_{i,0}(q) =& \frac{L_i(q)}{t_i}\Big(\frac{t_i(t_0-t_1)(t_i-t_2)}{L_i(q)^2(t_0+t_1+t_2) - 2L_i(q)(t_0t_1+t_1t_2+t_2t_0) + 3t_0t_1t_2}\Big)^{\frac{1}{2}}\,,
\end{aligned}
$$

Here $L_i(q)$ is the root of the following equation

$$
(\mathcal{L}-t_0)(\mathcal{L}-t_1)(\mathcal{L}-t_2) - q(3\mathcal{L})^3 = 0\,,
$$

with $\mathcal{L}|_{q=0} = t_i$.

From the equation (5) and (7), we can prove the series

$$
\mathbb{S}_i(1) = \mathbb{S}(1)|_{H=t_i},\ \mathbb{S}_i(H) = \mathbb{S}(H)|_{H=t_i},\ \mathbb{S}_i(H^2) = \mathbb{S}(H^2)|_{H=t_i}
$$

have the following asymptotic expansions;

$$\mathbb{S}_i(1) = e^{\frac{\mu_i}{z}}\Big(R_{i,00} + R_{i,01}z + R_{i,02}z^2 + \dots\Big),$$
$$\mathbb{S}_i(H) = e^{\frac{\mu_i}{z}}\Big(R_{i,10} + R_{i,11}z + R_{i,12}z^2 + \dots\Big),$$
$$\mathbb{S}_i(H^2) = e^{\frac{\mu_i}{z}}\Big(R_{i,20} + R_{i,21}z + R_{i,22}z^2 + \dots\Big).$$

Denote by D the differential operator $q\frac{d}{dq}$. From (5), we obtain the following result.

LEMMA 6. *We have*

$$R_{i,0n} = \frac{1}{C_0} R_{i,0}\,,$$
$$R_{i,1n} = \frac{1}{C_1}\Big((L_i - N_2)R_{i,0n} + \mathsf{D}R_{i,0\,n-1}\Big),$$
$$R_{i,2n} = \frac{1}{C_2}\Big((L_i - N_3)R_{i,1n} - N_4 R_{i,0n} + \mathsf{D}R_{i,1\,n-1}\Big).$$

Let $X = \frac{\mathsf{D}C_0}{C_0}$. The following equation was proven in [16],

$$X^2 - (L^3 - 1)X + \mathsf{D}X - \frac{2}{9}(L^3 - 1) = 0\,. \tag{8}$$

PROPOSITION 7. *We have*

$$\frac{R_{i,nk}}{R_{i,00}} \in \mathbb{Q}[t_0, t_1, t_2][C_0, C_0^{-1}, L_i, L^{-3}, X]\,.$$

Proof. The differential equations

$$\mathsf{D}L = \frac{L}{3}(L^3 - 1)\,,$$
$$\mathsf{D}L_i = \frac{L_i(L_i - t_0)(L_i - t_1)(L_i - t_2)}{(t_0 + t_1 + t_2)L^2 - 2(t_0t_1 + t_1t_2 + t_2t_0)L + 3t_0t_1t_2}$$

can be easily obtained from the definitions of L and L_i. Then the Proposition follows from equations (3),(4),(8) and Lemma 6. □

2.3. Graphs

For the genus g and the number of markings n in the stable range

$$2g - 2 + n > 0\,,$$

a decorated graph Γ consists of the data $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \mathsf{g}, \mathsf{p})$ such that

(a) V is the vertex set,
(b) E is the edge set (including self-edges),

(c) $\mathsf{N} : \{1, 2, \ldots, n\} \to \mathsf{V}$ is the marking assignment,
(d) $\mathsf{g} : \mathsf{V} \to \mathbb{Z}$ is a genus assignment with

$$g = \sum_{v \in \mathsf{V}} \mathsf{g}(v) + h^1(\Gamma)$$

and for which $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \mathsf{g})$ is stable,
(e) $\mathsf{p} : \mathsf{V} \to (\mathbb{P}^2)^{\mathsf{T}}$ is an assignment of a T-fixed point $\mathsf{p}(v)$ to each vertex $v \in \mathsf{V}$.

We denote by $\mathsf{G}_{g,n}$ the set of decorated graphs of genus g with n markings.

2.4. Localization formula

Here we apply the localization method to twisted Gromov-Witten theory of $\mathbb{P}^2$. Consider the moduli space

$$\overline{M}_{g,n}(\mathbb{P}^2, d)$$

of degree d stable maps of connected curves of genus g with n markings. Let

$$\pi : \overline{M}_{g,n}(\mathbb{P}^2, d) \to \overline{M}_{g,n}\,,\ \mathrm{ev}_i : \overline{M}_{g,n}(\mathbb{P}^2, d) \to \mathbb{P}^2\,,\ f : \mathcal{C} \to \mathbb{P}^2$$

be the universal structures. Define the twisted Gromov-Witten classes of $\mathbb{P}^2$ by

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n} := \pi_*\Big(e^{\mathsf{T}}(\mathbb{E}^\vee) \cup e^{\mathsf{T}}(R\pi_* f^* \mathcal{O}(3)) \cap [\overline{M}_{g,n}(\mathbb{P}^2, d)]^{\mathrm{vir},\mathsf{T}} \Big) \in H^*(\overline{M}_{g,n})\,.$$

We define the twisted Gromov-Witten series of $\mathbb{P}^2$ by

$$H_g^{\mathsf{T}}(\gamma_1, \ldots, \gamma_n) = \sum_{d=0}^{\infty} q^d \cdot \langle \gamma_1, \ldots, \gamma_n \rangle_{g,n} \in H^*(\overline{M}_{g,n}) \otimes \mathbb{Q}[[q]]\,.$$

Consider the forgetful map

$$p_m : \overline{M}_{g,n+m} \to \overline{M}_{g,n}\,.$$

For a power series $f(z) \in z^2\mathbb{Q}[[z]]$ with vanishing constant and linear terms, we define

$$\kappa(f) = \sum_{m=0}^{\infty} \frac{1}{m!} p_{m*}(f(\psi_{n+1}) \ldots f(\psi_{n+m})) \in H^*(\overline{M}_{g,n})\,.$$

Recall the cycle classes ϕ_i and ϕ^i in (2). We define a_{ij} by the following equation

$$\sum_i \phi_i \otimes \phi^i = \sum_{i,j} a_{ij} H^i \otimes H^j \,.$$

For a decorated graph $\Gamma \in \mathsf{G}_{g,n}$ and $(a_1, \dots, a_n) \in \mathbb{Z}^n$, we assign the following factors to vertex, leg and edge:

- For $v \in \mathsf{V}$, let

$$\kappa_v = \Big(\frac{P^2_{\mathsf{p}(v),00}}{e_{\mathsf{p}(v)}}\Big)^{1-g} \mathrm{Obs}_{\mathsf{p}(v)} \cdot \kappa\Big(z - z(\sum_{k=0}^{\infty} R_{\mathsf{p}(v),00} z^k)\Big)\,,$$

  where

$$\mathrm{Obs}_{\mathsf{p}(v)} = \Big(\prod_{i=1}^{g} \prod_{j \neq \mathsf{p}(v)} (t_{\mathsf{p}(v)} - t_j - c_i)\Big) \cdot \Big(\prod_{i=1}^{g} (3t_{\mathsf{p}(v)} - c_i)\Big) \cdot \Big(\prod_{i=1}^{g} (t_3 \mathsf{p}(v) - c_i)\Big)\,.$$

  with $c_1, \dots, c_g$ are Chern roots of the Hodge bundle,
- For $l \in \mathsf{L}$, let $B_l = R_{\mathsf{p}(v_l),a_l k} \psi_l^k$, where $v(l) \in \mathsf{V}$ is the vertex to which the leg is assigned.
- For $e \in \mathsf{E}$, let

$$\delta_e = \frac{\sum_{i,j} a_{ij} \Big(\sum_{k=0}^{\infty} R_{\mathsf{p}(v_1),ik} \psi_1^k\Big)\Big(\sum_{k=0}^{\infty} R_{\mathsf{p}(v_2),jk} \psi_2^k\Big)}{\psi_1 + \psi_2}$$

  where v_1,v_2 are the vertices adjacednt to the edge and ψ_1 and ψ_2 are the ψ-classes corresponding to the half-edges.

Applying the localization strategy to the series $H_g^{\mathsf{T}}(\gamma_1, \dots, \gamma_n)$ we obtain the following result.

Proposition 8. *We have*

$$H_g^{\mathsf{T}}(H^{a_1}, \dots, H^{a_n}) = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{Aut(\Gamma)} \Big[\Gamma, \prod_{v \in \mathsf{V}} \kappa_v \prod_{e \in \mathsf{E}} \delta_e \prod_{l \in \mathsf{L}} B_l\Big]\,.$$

3. Holomorphic anomlay equation

3.1. Proof of Theorem 1

Using the argument of [8, Section 2.3] we modify Theorem 8. Define the series $P_{i,nk}$ by the following equations

$$\sum_{k=0}^{\infty} P_{i,nk} z^k = \mathrm{Exp}\Big(-\sum_{k=1}^{\infty}\frac{N_{i,2k-1}}{2k-1} z^{2k-1}\Big)\Big(\sum_{k=0}^{\infty} R_{i,nk} z^k\Big)$$

where $N_{i,k} = \Big(-\frac{1}{3t_i}\Big)^k + \sum_{j\neq i}\Big(\frac{1}{t_i-t_j}\Big)^k$.

We obtain the equality

$$H_g^{\mathsf{T}}(H^{\alpha_1},\ldots,H^{\alpha_n}) = \sum_{\Gamma\in \mathsf{G}_{g,n}} \frac{1}{\mathrm{Aut}(\Gamma)}\Big[\Gamma, \prod_{v\in\mathsf{V}}\widetilde{\kappa}_v \prod_{e\in\mathsf{E}}\widetilde{\delta}_e \prod_{l\in\mathsf{L}}\widetilde{B}_l\Big], \tag{9}$$

with

- for $v\in\mathsf{V}$, $\widetilde{\kappa}_v = \Big(\frac{P^2_{\mathsf{p}(v),00}}{e_{\mathsf{p}(v)}}\Big)^{1-g}\Big(\prod_{i=1}^{g}(t_3-c_i)\Big)\kappa\Big(z-z(\sum_{k=0}^{\infty}P_{\mathsf{p}(v),00}z^k)\Big)$, where c_i are Chern roots of the Hodge bundle,
- for $l\in\mathsf{L}$, $\widetilde{B}_l = P_{\mathsf{p}(v_l),\alpha_l k}\psi_l^k$, where $v(l)\in\mathsf{V}$ is the vertex to which the leg is assigned,
- for $e\in\mathsf{E}$,

$$\widetilde{\delta}_e = \frac{\sum_{i,j} a_{ij}\Big(\sum_{k=0}^{\infty}P_{\mathsf{p}(v_1),ik}\psi_1^k\Big)\Big(\sum_{k=0}^{\infty}P_{\mathsf{p}(v_2),jk}\psi_2^k\Big)}{\psi_1+\psi_2}$$

  where v_1,v_2 are the vertices adjacednt to the edge and ψ_1 and ψ_2 are the ψ-classes corresponding to the half-edges.

LEMMA 9. *We have*

$$P_{i,nk}\in\mathbb{Q}[t_0,t_1,t_2][C_0,C_0^{-1},L_i,L^{-3},X].$$

Moreover, if we consider $P_{i,nk}$ as polynomials in L_i, $P_{i,nk}$ and $P_{j,nk}$ are same polynomials for all $0\le i,j\le 2$.

LEMMA 10. *We have*

$$\widetilde{\delta}_e\in\mathbb{Q}[t_0,t_1,t_2][L_{\mathsf{p}(v_1)},L_{\mathsf{p}(v_2)},L^{-3},X].$$

Recall $L_i(q)$ are roots of the equation

$$(\mathcal{L}-t_0)(\mathcal{L}-t_1)(\mathcal{L}-t_2)-q(3\mathcal{L})^3=0.$$

If $f(x_0,x_1,x_2)\in\mathbb{Q}[t_0,t_1,t_2][x_0,x_1,x_2]$ is a symmetric polynomial with respect to x_0,x_1,x_2, we have

$$f(L_0,L_1,L_2)\in\mathbb{Q}[t_0,t_1,t_2][L^3].$$

Therefore if we apply Theorem 5 to our case, Theorem 1 follows from (9), Lemma 9 and Lemma 10.

3.2. Proof of Theorem 2

We rewrite equation (9)

$$H_g^{\mathsf{T}}(H^{\alpha_1},\dots,H^{\alpha_n}) = \sum_{\gamma\in\mathsf{G}_{g,n}} \mathrm{Cont}_\Gamma\,.$$

Let $\Gamma\in\mathsf{G}_{g,n}$ be a decorated graph. Denote by F be the set of half-edges. From equation (9), we have

$$\mathrm{Cont}_\Gamma = \frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}^{\mathsf{F}}_{>0}}\prod_{v\in\mathsf{V}}\mathrm{Cont}^{\mathsf{A}}_\Gamma(v)\prod_{e\in\mathsf{E}}\mathrm{Cont}^{\mathsf{A}}_\Gamma(e)\prod_{l\in\mathsf{L}}\mathrm{Cont}^{\mathsf{A}}_\Gamma(l)\,,$$

where the vertex, edge and leg contributions with incident flag A-values $(a_1,\dots,a_n)$ and (b_1,b_2) and (c) respectively are

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(v) = \Big[\kappa\Big(z - z(\sum_{k=0}^{\infty}P_{\mathsf{p}(v),00}z^k)\Big)\Big]_{\prod_{i=1}^n\psi_i^{a_i-1}}\,,$$

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(v) = \Big[a_{ij}\Big(\sum_{k=0}^{\infty}P_{\mathsf{p}(v_1),ik}\psi_1^k\Big)\Big(\sum_{k=0}^{\infty}P_{\mathsf{p}(v_2),jk}\psi_2^k\Big)\Big]_{\sum_{k=0}^{b_2-1}(-1)^k\psi_1^{b_1+k}\psi_2^{b_2-1-k}}\,,$$

$$\mathrm{Cont}^{\mathsf{A}}_\Gamma(l) = P_{\mathsf{p}(v_l),\alpha_l c}\psi_l^c\,.$$

In the above expression, the subscript signifies a signed sum of the respective coefficients. Fix an edge $f\in\mathsf{E}(\Gamma)$:

(i) if Γ is connected after deleting f, denote the resulting graph by

$$\Gamma^0_f\in\mathsf{G}_{g-1,n+2}\,,$$

(ii) if Γ is disconnected after deleting f, denote the resulting two graphs by

$$\Gamma^1_f\in\mathsf{G}_{g_1,n_1+1}\ \text{ and }\ \Gamma^2_f\in\mathsf{G}_{g_2,n_2+1}$$

where $g=g_1+g_2$ and $n=n_1+n_2$.

Suppose f connect the T-fixed points $p_i,p_j\in\mathbb{P}^2$. Let the A-values of the corresponding half-edges be (k,l). By Lemma 6 we have

$$\frac{\partial\mathrm{Cont}^{\mathsf{A}}_\Gamma(f)}{\partial X} = \frac{(-1)^{k+l}3}{L}R_{\mathsf{p}(v_1),1k-1}R_{\mathsf{p}(v_2),1l-1}\,.$$

(i) If Γ is connected after deleting f, we have

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}^{\mathsf{F}}_{\geq 0}}\Big(\frac{L^3}{3C_1^2}\Big)\frac{\partial \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(f)}{\partial X}\prod_{v\in\mathsf{V}}\mathrm{Cont}^{\mathsf{A}}_{\Gamma}(v)\prod_{e\in\mathsf{E},\mathsf{e}\neq\mathsf{f}}\mathrm{Cont}^{\mathsf{A}}_{\Gamma}(e)$$
$$=\frac{1}{2}\mathrm{Cont}_{\Gamma^0_f}(H,H)\,.$$

(ii) If Γ is disconnected after deleting f, we obtain

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}^{\mathsf{F}}_{\geq 0}}\Big(\frac{L^3}{3C_1^2}\Big)\frac{\partial \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(f)}{\partial X}\prod_{v\in\mathsf{V}}\mathrm{Cont}^{\mathsf{A}}_{\Gamma}(v)\prod_{e\in\mathsf{E},\mathsf{e}\neq\mathsf{f}}\mathrm{Cont}^{\mathsf{A}}_{\Gamma}(e)$$
$$=\frac{1}{2}\mathrm{Cont}_{\Gamma^1_f}(H)\mathrm{Cont}_{\Gamma^2_f}(H)$$

The above two equations for all the edges of all the graphs $\Gamma\in\mathsf{G}_{g,n}$ explain the first two terms on the right-hand side of the equation in Theorem 2.

Similar argument for the leg $l\in\mathsf{L}(\Gamma)$ yields the last term in the equation of Theorem 2 from the following observation

$$\frac{\partial P_{i,0k}}{\partial X}=0\,,$$
$$\frac{\partial P_{i,1k}}{\partial X}=-\frac{C_0^2}{L^3}P_{i,0\,k-1}\,,$$

which can be easily obtained by the first two equations in Lemma 6.

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